

# Math 254A Lecture 5 Notes

Daniel Raban

April 7, 2021

## 1 Eventual Finiteness of $s_n(U)$ and Point Function Conditions

### 1.1 Recap

From last time, we have a  $\sigma$ -finite measure space  $(M, \lambda)$ , a locally convex topological vector space  $X$ , and a measurable map  $\varphi : M \rightarrow X$ . We also let  $U$  be the convex open subsets of  $X$ . In this case, the equivalent of type classes is  $T_n(U) = \{p \in M^n : \frac{1}{n} \sum_{i=1}^n \varphi(p_i) \in U\}$ , and we may let  $s_n(U) := \log \lambda^{\times n}(T_n(U))$ . We have shown that  $T_{n+m}(U) \supseteq T_n(U) \times T_m(U)$ , which implies that  $s_{n+m}(U) \geq s_n(U) + s_m(U)$  (taking values in  $[-\infty, \infty]$ ), and so, by Fekete,

$$s(U) = \lim_n \frac{s_n(U)}{n} = \sup_n \frac{s_n(U)}{n},$$

provided we show that either  $s_n(U) = -\infty$  or  $s_n(U) > -\infty$  for all sufficiently large  $n$ .

### 1.2 Eventual finiteness of $s_n(U)$

**Lemma 1.1.** *Either  $s_n(U) = -\infty$  or  $s_n(U) > -\infty$  for all sufficiently large  $n$ .*

*Proof.* Suppose  $s_m(U) > -\infty$ , i.e.  $\lambda^{\times m}(T_m(U)) > 0$ . Then  $T_{km}(U) \supseteq T_m(U)^k$ , so  $s_{km}(U) > -\infty$ . We need to control the indices in between.

Step 1: Reduce to the case where  $U \ni 0$ .<sup>1</sup> To do this, let  $x \in U$  and now consider  $\varphi'(m) = \varphi(m) - x$ . Then  $U' = U - x$  is a neighborhood of 0, and  $\{p : \frac{1}{n} \sum_{i=1}^n \varphi'(p_i) \in U'\} = \{p : \frac{1}{n} \sum_{i=1}^n \varphi(p_i) \in U\}$ .

Step 2: Since  $U$  is convex and  $U \ni 0$ ,  $tU \subseteq U$  for all  $t \in [0, 1]$ . Also, since  $U$  is open,  $U = \bigcup_{0 \leq t < 1} t \cdot U = \bigcup_{r \in \mathbb{N}} \frac{r}{r+1}U$ ; this is because  $x \in U$  implies there is some  $r \in \mathbb{N}$  such that  $\frac{r+1}{r}x \in U$ , i.e.  $x \in \frac{r}{r+1}U$ . The countable union is for measure theory purposes. So  $T_n(U) = \bigcup_r T_n(\frac{r}{r+1}U)$ , and so

$$\lambda^{\times n}(U) = \lim_{r \rightarrow \infty} \lambda^{\times n} \left( T_n \left( \frac{r}{r+1}U \right) \right).$$

---

<sup>1</sup>This step is not strictly necessary, but it makes our notation easier.

So there exists some  $r \in \mathbb{N}$  such that

$$\lambda^{\times m} \left( T_m \left( \frac{r}{r+1} U \right) \right) > 0.$$

Step 3: On the other hand,  $X = \bigcup_{q \in \mathbb{N}} q \cdot U$ , so for all  $n$ , we have  $\lambda^{\times n}(T_n(q \cdot U)) > 0$  for some  $q$ .

Step 4: Let  $n \gg m$  with  $n = km + \ell$  with  $\ell \in \{0, \dots, m-1\}$ . Suppose  $p \in M^n$ . Then

$$\begin{aligned} \frac{1}{n} \sum_{i=1}^n \varphi(p_i) &= \frac{1}{n} \left( \sum_{i=1}^n \varphi(p_i) + \sum_{i=m+1}^{2m} \varphi(p_i) + \dots + \sum_{i=(k-1)m+1}^{km} \varphi(p_i) + \sum_{i=km+1}^n \varphi(p_i) \right) \\ &= \frac{m}{n} \left( \frac{1}{m} \sum_{i=1}^n \varphi(p_i) + \frac{1}{m} \sum_{i=m+1}^{2m} \varphi(p_i) + \dots + \frac{1}{m} \sum_{i=(k-1)m+1}^{km} \varphi(p_i) \right) \\ &\quad + \underbrace{\frac{\ell}{n} \cdot \frac{1}{\ell} \sum_{i=km+1}^n \varphi(p_i)}_*. \end{aligned}$$

For each of these  $k$  terms, we have positive measure for the event that  $\frac{1}{m} \sum_{i=*}^{*+m} \varphi(p_i) \in \frac{r}{r+1} U$ . Hence, we have positive measure that  $\frac{1}{k} \left( \frac{1}{m} \sum_{i=1}^m \varphi(p_i) + \dots + \frac{1}{m} \sum_{i=(k-1)m+1}^{km} \varphi(p_i) \right) \in \frac{r}{r+1} U$  (and we can even replace this by  $\frac{mk}{n}$  times this). By step 3, we have positive measure that  $* \in q \cdot U$  for some  $q$  independent of  $n$  and hence  $\frac{\ell}{n} \cdot * \in \frac{q\ell}{n} U$ . If all of these positive measure events occur, then

$$\frac{1}{n} \sum_{i=1}^n \varphi(p_i) \in \frac{r}{r+1} \cdot U + \frac{q\ell}{n} U.$$

Provided  $n \geq q \cdot \ell \cdot (r+1)$ , this implies

$$\frac{1}{n} \sum_{i=1}^n \varphi(p_i) \in \frac{r}{r+1} U + \frac{1}{r+1} U = U.$$

Hence,  $s_n(U) > -\infty$  for this  $n$ . □

**Remark 1.1.** It is also possible that  $\lambda^{\times n}(T_n(U)) = +\infty$ , so  $s_n(U) = +\infty$ , and we may get  $s(U) = +\infty$ . Fekete's lemma still works, but the result is not meaningful. You usually want to look for additional reasons of why  $s$  is locally finite. The simplest condition is that if  $\lambda(M) < \infty$ , then  $\lambda^{\times n}(T_n(U)) \leq \lambda(M)^n$  for all  $n$ .

### 1.3 Checking conditions to extend $s$ to a point function

Next, we want to switch to point functions  $s(x) = \inf\{s(U) : U \in \mathcal{U}, U \ni x\}$ .

**Proposition 1.1.** *Under the same conditions as before,  $s$  is concave.*

*Proof.*  $T_{n+m}(U) \supseteq T_n(U) \times T_m(U)$ . Similarly, let  $x \in T_n(U)$  and  $y \in T_m(V)$  (where  $U, V \in \mathcal{U}$ ). Then the concatenation  $z = xy$  satisfies

$$\frac{1}{2n} \sum_{i=1}^{2n} \varphi(z_i) = \frac{1}{2} \left( \frac{1}{n} \sum_{i=1}^n \varphi(x_i) + \frac{1}{n} \sum_{i=1}^n \varphi(y_i) \right) \in \frac{1}{2}U + \frac{1}{2}V.$$

So  $T_{2n}(\frac{1}{2}U + \frac{1}{2}V) \supseteq T_n(U) \times T_n(V)$ , which tells us that

$$\frac{s_{2n}(\frac{1}{2}U + \frac{1}{2}V)}{2n} \geq \frac{1}{2} \left( \frac{s_n(U)}{n} + \frac{s_n(V)}{n} \right).$$

After letting  $n \rightarrow \infty$ , we get

$$s \left( \frac{1}{2}U + \frac{1}{2}V \right) \geq \frac{1}{2}(s(U) + s(V)).$$

By a previous lemma (the argument with dyadic rationals and applying upper semicontinuity), this gives that the point function  $s(x)$  is concave.  $\square$

Next, we quickly check that condition (S1) holds: If  $U \subseteq U_1 \cup \dots \cup U_k$ , then  $T_n(U) \subseteq T_n(U_1) \cup \dots \cup T_n(U_k)$ . Using subadditivity and taking logs, we get

$$\frac{s_n(U)}{n} \leq \frac{\log K}{n} + \max_i \frac{s_n(U_i)}{n},$$

which gives

$$s(U) \leq \max_i s(U_i).$$

We also need conditions under which we can check (S2):  $s(U) = \sup\{s(K) : K \subseteq U, K \text{ compact}\}$ , where  $s(K) = \inf\{\max_i s(U_i) : K = U_1 \cup \dots \cup U_k, U_i \in \mathcal{U}\} = \sup_{x \in K} s(x)$  (by a previous lemma). To deduce (S2) in the setting of generalized type-counting, we need to assume:

Every open convex set  $U$  can be written as a countable union of compact, convex sets.

**Example 1.1.** In  $\mathbb{R}^d$ , by intersecting with balls, we can write every  $U$  as a countable union of bounded, open, convex sets, and then we can express each of these as a countable union of compact convex sets by looking at the set of points under a certain distance from the boundary.

**Example 1.2.** If  $X = Y^*$  with the weak\*-topology, this property also holds, but we will show this later.